

Intermediate inflation under the scrutiny of recent data

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Abstract

We use the flow equations to determine the different hierarchy Hubble parameters as a function of the number of e-folds for intermediate models in single-field inflation. The obtained expressions allow us to determine at second order in the hierarchy Hubble parameters different observational parameters. We distinguish the scalar spectral index, its running and the tensor-to-scalar ratio, among others. Recently, it has been noticed that measurements released by Planck, combined with the WMAP large-angle polarization are in tension with this sort of model. Here, we show in detail why this occur. The conclusions do not change even when the recent BICEP2 data are included.

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I. INTRODUCTION

In single-field inflationary universe theories there exist three ways to address their study. One of them is the usual slow-roll approach which has been considered by many researcher from its beginning[1–9]. This approximation puts generic restrictions on the effective inflaton scalar potential[10], but it has been quite successful in describing the main characteristics of inflationary scenarios. However, it has been shown that the slow-roll approximation is invalid for all models in which the scalar field interacts with itself[11, 12], specifically when the potential presents a inflexion point, where a violation of the slow roll approximation is encountered. In this approach the expansion of the universe is governed by the effective scalar field potential, $V(\phi)$ (ϕ here is the *scalar inflaton field*), where the kinetics term is much smaller than the potential energy. Added to this approximation it is also assumed that $|\frac{d \ln \phi}{dt}| \ll H$, which simplifies the Klein-Gordon equation to $3H\dot{\phi} \approx \frac{dV(\phi)}{d\phi}$, where dots represent derivatives with respect to the cosmological time t . To make this scheme to work it is needed to provide an explicit expression for the effective inflaton potential $V(\phi)$. The picture here it is that during inflation the inflaton field slowly rolls down to the minimum of the scalar potential.

Essentially, we can observe that the slow-roll inflationary scenario is intimately related to the inflaton scalar field potential, $V(\phi)$. Since, most of the observational quantities (such that the spectral scalar index, n_s , the running scalar index, $\alpha_s \equiv \frac{dn_s}{d \ln k}$, the tensor-to-scalar ratio, r , the spectral tensor index, n_T , etc.) are expressed in terms of the scalar potential and its derivatives, via the different slow-roll parameters. Therefore, we expect that the observational data will put strong constraint on the shape of the potential. In fact, one of the predictions of the slow-roll inflation is that the form of the potential should be extremely flat and smooth in the case of minimally coupled scalar field. Due to this, it is possible that the slow-roll approximation may breaks down for some range in the values of the inflaton scalar field where the scalar potential does not present these qualities. Moreover, it has been noticed that in hybrid inflation the slow-roll approximation breaks down at all points in the evolution of the scalar field[13, 14]. More dramatic situations can occur. For instance, in the model discussed by Starobinsky[15], characterized by a scalar potential that presents a sudden change in its slope at $\phi = \phi_0$, where the inflaton potential becomes
$$V(\phi) = \begin{cases} V_0 + A_+(\phi - \phi_0) & \text{for } \phi > \phi_0 \\ V_0 + A_-(\phi - \phi_0) & \text{for } \phi < \phi_0 \end{cases}.$$
 The change in the slope of this potential may be sufficiently abrupt so that the slow-roll approximation can be violated, and for $A_+ > A_- > 0$ the field enters in a "fast-roll" solution, characterized by the expression $\ddot{\phi} = -3H\dot{\phi}$ [16, 17].

Intermediate and power law inflations provide exact solutions for the time evolution of cosmological perturbations, and inflation can occur although the slow-roll conditions are violated. It is therefore interesting to investigate the exact predictions out of the slow-roll predictions. For instance, in [18] was found that the consistency relation obtained from the slow-roll approximation, $C_2^T/C_2^S \approx -6.93n_T$, may differ considerably from the exact result. In power law inflation, where the scale factor evolves as $a \propto t^p$, with $p > 1$, the exact and the slow-roll results differ by a factor $F(n_T)/(1 - n_T/2)$, where $F(n_T)$ denotes a numerical integration. For instance, if p is taken to be two the error is found to be as high as a 34%.

Another way to study inflationary universe models out of the slow-roll approximation is given by the Hamilton-Jacobi approach[19–23]. In this schema the basic quantity results to be the Hubble parameter which is given in terms of the inflaton field, $H(\phi)$, the so called generating function[24]. In this approach the form of the potential is deduced, and since we are out of the slow-roll approximation, application to the final period of inflation is possible. In this period the kinetic term associated to the inflaton field in the Friedmann equation becomes important when compared with the scalar potential. In this way, this approach becomes useful when studying the final stage of inflation in which the reheating phase occurs. In the same way, this approach can be applicable to models that cannot be studied within the slow-roll approximation, for instance in models where the problem of a large slow-roll parameter η of some supersymmetry-inspired inflationary models arises, and it can be applied to small scales that leave the horizon at times close to the end of inflation, where, as we have mentioned, the slow-roll approximation necessarily breaks down. In this approach we can take the inflaton scalar field, ϕ , as a *time variable*, and for that, we claim that this field increases monotonically, i.e. its time derivative, $\dot{\phi}$, should not change of sign along the inflationary phase.

The third way of studying inflationary universe models is through the introduction of certain parameters which are subtended by a sort of order imposed on them[25, 26]. Each of these hierarchy parameters is characterized by its dependence on the order of the scalar field derivative of the Hubble ratio, $H(\phi)$. Prior to the introduction of the approach as such let us bring out the so-called *first Hubble hierarchy parameter* defined by

$$\epsilon_H \equiv -\frac{d \ln H}{d \ln a} = \left(\frac{m_{Pl}^2}{4\pi} \right) \left(\frac{1}{H} \frac{dH}{d\phi} \right)^2, \quad (1)$$

where m_{Pl} represents the Planck mass.

The previous fundamental quantity becomes defined from the the acceleration equation for the

scale factor

$$\frac{\ddot{a}}{a} = H^2 [1 - \epsilon_H]. \quad (2)$$

We observe that during inflation, i.e., when \ddot{a} is positive, this parameter satisfies the bound $\epsilon_H \leq 1$, where the equality is obtained at the end of the inflationary period.

As we did with the previous definition we can introduce similar parameters which are also called *Hubble hierarchy parameters*. The *second Hubble hierarchy parameter*, η_H , is defined by

$$\eta_H \equiv - \left(\frac{d \ln \left(\frac{dH}{d\phi} \right)}{d \ln a} \right) = \frac{m_{Pl}^2}{4\pi} \left[\frac{1}{H} \frac{d^2 H}{d\phi^2} \right]. \quad (3)$$

Similarly, we can introduce the *third Hubble hierarchy parameter*, ξ_H^2 , defined by

$$\xi_H^2 \equiv \left(\frac{d \ln \left(\frac{dH}{d\phi} \right)}{d \ln a} \right) \left(\frac{d \ln \left(\frac{d^2 H}{d\phi^2} \right)}{d \ln a} \right) = \left(\frac{m_{Pl}^2}{4\pi} \right)^2 \left[\frac{1}{H^2} \frac{dH}{d\phi} \frac{d^3 H}{d\phi^3} \right]. \quad (4)$$

We can extend these definitions to higher derivatives of the Hubble parameter, so that we can define in general

$$\begin{aligned} {}^l\lambda_H &\equiv (-)^l \left(\frac{d \ln \left(\frac{dH}{d\phi} \right)}{d \ln a} \right) \left(\frac{d \ln \left(\frac{d^2 H}{d\phi^2} \right)}{d \ln a} \right) \dots \dots \dots \left(\frac{d \ln \left(\frac{d^l H}{d\phi^l} \right)}{d \ln a} \right) \\ &= \left(\frac{m_{Pl}^2}{4\pi} \right)^l \left[\frac{1}{H^l} \left(\frac{dH}{d\phi} \right)^{l-1} \frac{d^{l+1} H}{d\phi^{l+1}} \right], \quad (l \geq 1), \end{aligned} \quad (5)$$

where we see that for $l = 1$ it gives ${}^1\lambda_H \equiv \eta_H$ and for $l = 2$ it provides ${}^2\lambda_H \equiv \xi_H^2$, etc.

In this approach the set of equations is based on derivatives of the different Hubble hierarchy parameters ${}^l\lambda_H$ with respect to the *e-folds number*, N . Here, the parameter N is interpreted as the number of e-folds before inflation ends. This quantity is defined as $N(t) \equiv \ln \frac{a(t_{end})}{a(t)}$, where $a(t_{end})$ represents the scale factor evaluated at the end of inflation. Usually this quantity is written as $N = - \int_t^{t_{end}} H dt$.

In this paper we would like to study intermediate inflationary universe models by taking into account the hierarchy flow equations. After giving a brief introduction to intermediate inflation, we solve the set of hierarchy equations in such a way that, as we will see, it will be necessary to determine the *first Hubble hierarchy parameter*, ϵ_H , since the other high parameters will be determined from this one. After doing this, we will determine different observational quantities in terms of these parameters. We will obtain these quantities at second order of the Hubble hierarchy parameters. Then, we will contrast these parameters with the corresponding observational data

released by Planck, together with the WMAP large-angle polarization observations[27]. Interesting results the data expressed in the n_s/r plane, where the different models make predictions on it. Certain increasing precision will help in selecting the appropriated model. On the other hand, very recently, the ground based BICEP2 experiment[28] announced the detection for the first time of primordial B-mode polarization on the cosmic microwave background radiation. This significant measurement implies that the tensor-to-scalar ratio, r , present a non-zero value at seven sigma, whose value (combination of Planck + WMAP9 + HighL + BICEP2) results to be $r = 0.2^{+0.07}_{-0.05}$. Not only this discovery should be confirmed by another ground based or satellite experiment, but also this new experiment should determine the amplitude of the corresponding gravitational wave signal. Perhaps, Planck satellite will help on this task. As a result of this finding the n_s/r plane becomes modified. In this article we would like to use the current available cosmological data to look at the feasibility of the inflationary model called intermediate.

II. A BRIEF INTRODUCTION TO INTERMEDIATE INFLATION

Intermediate inflationary universe models was introduced as an exact solution for a particular scalar field potential of the type $V(\phi) \propto \phi^{-4(f^{-1}-1)}$ [29], where f is a free parameter which ranges as $0 < f < 1$. With this sort of potential it is possible in the slow-roll approximation to have a spectrum of density perturbations which presents an exact scale-invariant spectral index, i.e. $n_s = 1$, the so-called Harrison-Zel'dovich spectrum of density perturbations.

The main motivation to study intermediate inflationary model becomes from string/M theory. This theory suggests that in order to have a ghost-free action high order curvature invariant corrections to the Einstein-Hilbert action must be proportional to the Gauss-Bonnet (GB) term [30]. GB terms arise naturally as the leading order of the expansion to the low-energy string effective action, where is the inverse string tension[31]. This kind of theory has been applied to possible resolution of the initial singularity problem[32], to the study of Black-Hole solutions[33], accelerated cosmological solutions[34], among others. In particular, it has been found that for a dark energy model the GB interaction in four dimensions with a dynamical dilatonic scalar field coupling leads to a solution of the form $a(t) = a_0 \exp \left[\left(\frac{2}{\kappa n} \right) t^{\frac{1}{2}} \right]$ [35]. Here, $\kappa = 8 \pi G$ and n is an arbitrary constant. Actually, this kind of behavior of the scale factor is what characterizes the intermediate inflation, in which the scale factor evolves as

$$a(t) = a_0 \exp \left(At^f \right), \quad (6)$$

where A is a positive constant and f was introduced above. In this way, the expansion of the universe is slower than standard de-Sitter inflation ($a(t) = \exp(Ht)$), but faster than power law inflation ($a(t) = t^p; p > 1$). Thus, the idea that inflation, or specifically, intermediate inflation, comes from an effective theory at low dimension of a more fundamental string theory is in itself very appealing. Thus, in brane universe models the effective theories that emerge from string/M theory lead to a Friedmann equation which is proportional to the square energy density, on the one hand, and an evolving intermediate scale factor, in addition, it makes interesting to study their mixture by itself, i.e., an intermediate inflationary universe model in a brane world effective theory. In it was shown that the combination $n_s = 1$ and $r > 0$ is given by a version of the intermediate inflation model in which the scale factor varies as $a \sim \exp(\frac{t}{t_o})^{2/3}$ in which the slow-roll approximation was used[36]. In the same reference the status of this model was evaluated in general terms in light of the WMAP3 data. But, as was described in the introduction, new data are accessible nowadays, and we pretend to use them in order to appraise intermediate inflation.

In the following we will describe the main results that occur in intermediate inflation in the most simpler model in which a single scalar inflaton field is present.

Our basic equations are the Friedmann equation which we take to be

$$H^2 = \left(\frac{8\pi}{3m_{Pl}^2} \right) \left[\frac{1}{2} \dot{\phi}^2 + V(\phi) \right], \quad (7)$$

together with the energy density conservation equation

$$\ddot{\phi} + 3H\dot{\phi} = -\frac{dV(\phi)}{d\phi}. \quad (8)$$

It is possible to find exact inflationary universe solutions from this set of equations by giving the functional form of the Hubble parameter in term of the inflaton field, i.e. $H(\phi)$ by making use of the Hamilton-Jacobi approach. From now on, we will adopt this type of approach.

From expressions (7) and (8) yield to

$$\dot{\phi}^2 = - \left(\frac{m_{Pl}^2}{4\pi} \right) \frac{dH}{dt}. \quad (9)$$

Now, equation (6) allows to obtain the Hubble factor as a function of the cosmological time, and thus the latter equation can be integrated for obtaining the scalar field as a function of time. The result is given by

$$\phi(t) = \left(\frac{m_{Pl}^2}{4\pi} \beta A t^f \right)^{\frac{1}{2}}, \quad (10)$$

where $\beta \equiv 4(f^{-1} - 1)$ is a dimensionless positive constant. This latter expression can be inverted to obtain the cosmological time as a function of the scalar field which can be used for obtaining the generating function. We get that in this case the generating function becomes

$$H(\phi) = \left(\frac{m_{Pl}^2}{4\pi} \beta A \right)^{\frac{\beta}{4}} A f \phi^{-\frac{\beta}{2}}. \quad (11)$$

From this latter expression we can get the scalar inflaton potential, which becomes expressed by

$$V(\phi) = \frac{3m_{Pl}^2}{8\pi} H^2 \left[1 - \frac{m_{Pl}^2}{12\pi} \left(\frac{1}{H} \frac{dH}{d\phi} \right)^2 \right] = \mathcal{V}_0 \phi^{-\beta} \left[1 - \left(\frac{m_{Pl}^2}{48\pi} \right) \left(\frac{\beta}{\phi} \right)^2 \right], \quad (12)$$

where the constant \mathcal{V}_0 becomes given by $\mathcal{V}_0 \equiv \frac{3}{2} \left(\frac{m_{Pl}^2}{4\pi} \right)^{\frac{\beta}{2}+1} f^2 \beta^{\frac{\beta}{2}} A^{\frac{2}{f}}$.

At this point some comments are in order. First, the potential presents extreme values at $\phi_{\pm} = \pm \sqrt{\left(1 + \frac{2}{\beta}\right) B}$, with $B = \left(\frac{m_{Pl}^2}{48\pi}\right) \beta^2$ and in the following we will analyze the scalar potential in the ranges $\phi > 0$ and $\phi < 0$, separately. Second, it is found that the second derivative of the potential evaluated at the previous extreme points becomes $\left. \frac{d^2 V(\phi)}{d\phi^2} \right|_{\phi=\phi_{\pm}} = -\frac{2\beta^2}{(2+\beta)B} \left[\pm \left(1 + \frac{2}{\beta}\right) B \right]^{-\frac{\beta}{2}}$. We see that $\phi = \phi_+$ represents an maximum extreme point of the scalar potential for any value of the β parameter. Therefore, no minimum it is found in the range in which the inflaton field is positive. The situation in which the point $\phi = \phi_-$ is concerned, is more subtle. Note here that it is possible to have an extreme value (maximum or minimum) depending of the value of $\beta/2$. Firstly, if this value is integer odd, then we find that the second derivative of the potential becomes positive and that the potential present a minimum at the point $\phi = \phi_-$. But, if the parameter $\beta/2$ is integer even again it is found a maximum point in the scalar potential. The situation in which this parameter becomes non-integer can not be since the second derivative of the potential becomes an imaginary number.

In light from the previous analysis we see that the negative range for the inflaton field becomes more restricted than the case in which the range becomes positive. In our analysis we would like to relax the β (or f) parameter, so that, in the following we take the inflaton field within the positive range. The price that we pay here is that, since the scalar potential does not present a minimum, then it is not possible to finish the inflationary period[37].

III. THE HIERARCHY FLOW EQUATIONS

The corresponding hierarchy flow equations become expressed as[26]

$$\begin{aligned}\frac{d\epsilon_H}{dN} &= \epsilon_H(\sigma + 2\epsilon_H), \\ \frac{d\sigma}{dN} &= -5\epsilon_H\sigma - 12\epsilon_H^2 + 2\xi_H^2, \\ \frac{d^l\lambda_H}{dN} &= \left[\frac{l-1}{2}\sigma + (l-2)\epsilon_H\right]^l\lambda_H + {}^{l+1}\lambda_H, \quad (l \geq 2)\end{aligned}\tag{13}$$

where σ is defined as $\sigma = 2\eta_H - 4\epsilon_H$.

As specified in the introduction is only necessary to know the first hierarchy parameter, ϵ_H , as a function of the number of e-folds, N , and then from it, the other parameters are determined by using the previous set of equations. So, let us assume for a moment that we know the parameter ϵ_H as a function of N . Then, from the first equation of the set (13) we get that

$$\eta_H = \epsilon_H + \frac{1}{2}(\ln \epsilon_H)',\tag{14}$$

where the prime represents a derivative with respect to the number of e-folds, N .

Similarly, from the second equation of the set (13) we obtain

$$\xi_H^2 = \epsilon_H^2 + \frac{3}{2}\epsilon_H' + \frac{1}{2}(\ln \epsilon_H)''.\tag{15}$$

Analogously, we get for $l = 3$ the following expression

$${}^3\lambda_H = \epsilon_H^3 + 3\epsilon_H\epsilon_H' - \frac{3}{4}\frac{1}{\epsilon_H}(\epsilon_H')^2 + \frac{3}{2}\epsilon_H'' + \frac{1}{2}\left(\epsilon_H - \frac{1}{2}(\ln \epsilon_H)'\right)(\ln \epsilon_H)'' + \frac{1}{2}(\ln \epsilon_H)''',\tag{16}$$

and similarly the other high parameters can be obtained. As we see, we can say that any of the high hierarchy parameters can be determined from the first Hubble hierarchy parameter, just using the set of flow equations.

IV. THE HIERARCHY HUBBLE PARAMETERS FOR INTERMEDIATE INFLATION

A. The generating function in terms of N

The generating function as a function of the number of e-folds we take to be given by

$$H(N) = H_e \left(1 - \frac{4N}{\beta}\right)^{-\frac{\beta}{4}},\tag{17}$$

where H_e represents the value of the Hubble parameter at the end of inflation (when $N = 0$) and is given by $H_e = (Af) \left[\frac{4A}{\beta} \left(\frac{m_{Pl}^2}{4\pi} \right)^2 \right]^{\frac{\beta}{4}}$. Since H must be a positive real quantity we need to impose the constraint $\beta > 4N$, which for $N = (50, 60, 70)$, the usual values of the required number of e-folds, implies that β should be greater than $(200, 240, 280)$. This is equivalent to taking the inequality $f < (0.019, 0.016, 0.014)$ for values of the parameter f .

Since we know how the scale factor evolves with the cosmological time, we get the Hubble parameter as a function of time. By using equation (17) we obtain the time as a function of N

$$t(N) = \left[\frac{\beta}{4A} \left(1 - \frac{4N}{\beta} \right) \right]^{\frac{1}{f}}. \quad (18)$$

Note that the period of inflation becomes given by $t_e \equiv t(N = 0) = \left(\frac{\beta}{4A} \right)^{\frac{1}{f}}$. On the other hand, expression (18) yields to

$$\phi(N) = \frac{\beta}{2} \sqrt{\left(\frac{m_{Pl}^2}{4\pi} \right) \left(1 - \frac{4N}{\beta} \right)}, \quad (19)$$

where $\phi_e \equiv \phi(N = 0) = \frac{\beta}{2} \sqrt{\frac{m_{Pl}^2}{4\pi}}$ is the value of the inflaton field at the end of inflation. It is not difficult to see that when equation (19) is substituted into equation (17) we obtain the generating function (11).

B. The hierarchy Hubble parameters in terms of N

We can use expression (17) for getting the first hierarchy parameter, ϵ_H , since it is given by $\epsilon_H = \frac{d}{dN} (\ln H(N))$. This becomes

$$\epsilon_H(N) = \left(\frac{1}{1 - \frac{4N}{\beta}} \right). \quad (20)$$

Note that at the end of inflation, i.e. when $N = 0$, this parameter takes the value one, as it should be.

The other parameters, η_H and ξ_H^2 , can be obtained by using expressions (14) and (15), respectively. They become

$$\eta_H(N) = \left(\frac{1 + \frac{2}{\beta}}{1 - \frac{4N}{\beta}} \right), \text{ and } \xi_H^2(N) = \frac{1 + \frac{6}{\beta} + \frac{8}{\beta^2}}{\left(1 - \frac{4N}{\beta} \right)^2}, \quad (21)$$

respectively. We will use these latter expressions in order to get some explicit expressions for some parameters related to scalar density perturbations and relic gravitational waves.

V. SCALAR AND TENSOR PERTURBATIONS

A. A general approach to quantum perturbations

Inflation causes perturbations through the amplification of quantum fluctuations, which are stretched to astrophysical scales by the accelerated expansion. Inflation generates two types of perturbations, the density perturbations, which come from quantum fluctuations in the inflaton field, together with the corresponding scalar metric perturbation[38], and relic gravitational waves which are tensor metric fluctuations[39]. The former gives rise to gravitational instability and acts as seed of structure formation[40], while the latter predicts a stochastic background of relic gravitational waves.

The gauge invariant Mukhanov-Sasaki variable[41–43], defined from $u = z\mathcal{R}$, with $z = a\frac{\dot{\phi}}{H}$ and \mathcal{R} corresponds to the gauge-invariant comovl curvature perturbation, plays an prominent role in cosmology, not only because of its simple relation with the perturbed scalar curvature on the spatial hypersurfaces but also because, in general relativity, as we will see, it behaves as a KleinGordon field. In terms of the Fourier transformed of the Mukhanov-Sasaki variable the corresponding equation results to be

$$\frac{d^2 u_k}{d\eta^2} + \left(k^2 - \frac{1}{z} \frac{d^2 z}{d\eta^2} \right) u_k = 0, \quad (22)$$

where $\eta = \int \frac{1}{a} dt$ is the conformal time and $\frac{1}{z} \frac{d^2 z}{d\eta^2}$ corresponds to the mass term.

We have that during inflation $k^2 \gg \frac{1}{z} \frac{d^2 z}{d\eta^2}$, and thus the latter equation can be solved to get at early time a solution of the type $u_k(\eta) \sim e^{-ik\eta} \left(1 + \frac{\mathcal{A}_k}{\eta} + \dots \right)$.

When $k^2 \ll \frac{1}{z} \frac{d^2 z}{d\eta^2}$, it is found that the physical modes present wavelengths much bigger than the curvature scale.

In solving equation (22), it is needed to impose boundary conditions. Usually, the asymptotic conditions are taken to be

$$u_k \rightarrow \begin{cases} \frac{1}{\sqrt{2k}} e^{-ik\eta} & \text{as } -k\eta \rightarrow \infty, \\ \mathcal{A}_k z & \text{as } -k\eta \rightarrow 0. \end{cases}, \quad (23)$$

the so-called Bunch-Davies vacuum state[44]. These conditions guarantee that perturbations that are generated well inside the horizon, i.e. in the region where $k \ll aH$, the modes approach plane waves, and those that are generated well outside the horizon, i.e. in the region where $k \gg aH$, are fixed.

The primordial scalar perturbation is defined from the two point correlation function, which results to be

$$\mathcal{P}_{\mathcal{R}}(k) = \frac{k^3}{2\pi^2} \langle \mathcal{R}_{\vec{k}}, \mathcal{R}_{\vec{k}} \rangle \delta(\vec{k}' + \vec{k}) = \frac{k^3}{2\pi^2} \left| \frac{u_k}{z} \right|^2. \quad (24)$$

From the primordial scalar perturbations we can define the scalar spectral index

$$n_s - 1 \equiv \frac{d \ln \mathcal{P}_{\mathcal{R}}}{d \ln k}, \quad (25)$$

which in terms of the hierarchy Hubble parameters and considering into account higher order corrections, this reduces to[45]

$$n_s - 1 = -4\epsilon_H + 2\eta_H - 2(1 + \mathcal{C})\epsilon_H^2 - \frac{1}{2}(3 - 5\mathcal{C})\epsilon_H\eta_H + \frac{1}{2}(3 - \mathcal{C})\xi_H^2, \quad (26)$$

where $\mathcal{C} = 4(\ln 2 + \gamma) - 5$ with $\gamma = 0.5772$ represents the *Euler-Mascheroni constant*.

In the same order correction the *running scalar spectral index*, $\alpha_s \equiv \frac{dn_s}{d \ln k}$ becomes

$$\alpha_s = \frac{1}{1 - \epsilon_H} \left[8\epsilon_H^2 + 10\epsilon_H\eta_H - 2\xi_H^2 - \frac{7\mathcal{C} - 9}{2}\epsilon_H\xi_H^2 + \frac{\mathcal{C} - 3}{2}\eta_H\xi_H^2 \right]. \quad (27)$$

On the other hand, transverse-traceless tensor perturbations are generated from quantum fluctuations during inflation[40]. Tensor perturbations do not couple to matter and thus they are determined by the dynamics of the background metric only. The tensor perturbations evolve like minimally coupled massless fields whose spectrum becomes represented by $\mathcal{P}_{\mathcal{T}}$ and thus, we can introduce the *gravitational wave spectral index* n_T given by $n_T \equiv \frac{d \ln \mathcal{P}_{\mathcal{T}}}{d \ln k}$, which becomes at second order correction[46]

$$n_T = -2\epsilon_H - (3 + \mathcal{C})\epsilon_H^2 + (1 + \mathcal{C})\epsilon_H\eta_H. \quad (28)$$

In the same way, we can introduce the *running tensor spectral index*, $\alpha_T \equiv \frac{dn_T}{d \ln k}$ which results into

$$\alpha_T = -4\frac{\epsilon_H}{1 - \epsilon_H}(\epsilon_H - \eta_H) - (1 + \mathcal{C})\frac{\epsilon_H\xi_H}{1 - \epsilon_H}. \quad (29)$$

We define the tensor-to-scalar amplitude ratio $r \equiv \frac{\mathcal{P}_{\mathcal{T}}}{\mathcal{P}_{\mathcal{R}}}$ which results to be at second order correction

$$r = 16\epsilon_H [1 + 2\mathcal{C}(\epsilon_H - \eta_H)]. \quad (30)$$

We want to determine these general expressions for the case of intermediate inflation, which we describe in the next subsection.

B. The intermediate inflationary case

We would like to apply the previous approach to the intermediate inflationary universe model, where our finality is to obtain the different expression as a function the number of e-folds in order to compare with the current observational data. In this context this approach is similar to that described in Refs.[47] and [48]. Thus, in doing this, we use expressions (20) and (21) in the different relevant expressions for n_s , α_s , n_T , α_T and r . Firstly, we obtain for the parameter $n_s(N)$ the following expression

$$n_s(N) - 1 = -2 \left(\frac{1}{1 - \frac{4N}{\beta}} \right) \left[1 - \frac{2}{\beta} + \left(1 - (3 + \mathcal{C}) \frac{1}{\beta} - 2(3 - \mathcal{C}) \frac{1}{\beta^2} \right) \left(\frac{1}{1 - \frac{4N}{\beta}} \right) \right]. \quad (31)$$

We can use this expression to obtain the corresponding value of the parameter f when the current data released by planck[27], which is $n_s = 0.9603 \pm 0.0073$, is considered at some given e-folds number, N . For instance, if we take $N = 60$ we get $f = 0.03181 \pm 0.0005$, and if $N = 70$ we obtain $f = 0.02741 \pm 0.00006$. These values for f correspond to $\beta \simeq 122$ and $\beta \simeq 142$, respectively. However, we saw previously that in order to have the Hubble parameter real it is needed to satisfy that β must be greater than $4N$ (see expression (17)), and we observe that these value for β violate this condition, therefore, we can conclude that there not exist value of f which agrees with the value of n_s released by Planck together with an appropriated value of the number of e-folds.

From the previous expression (31) we can obtain the running scalar spectral index, which becomes

$$\alpha_s(N) = - \frac{\left(\frac{\beta}{4N} - 1 \right)}{\left(1 - \frac{4N}{\beta} \right)^2} \left[8 \left(2 + \frac{1}{\beta} - \frac{2}{\beta^2} \right) - \left(1 + \frac{6}{\beta} + \frac{8}{\beta^2} \right) (3 - \mathcal{C} - 2\beta(3 - 2\mathcal{C})) \left(\frac{1}{1 - \frac{4N}{\beta}} \right) \right]. \quad (32)$$

Planck[27] satellite reported a value for this parameter given by $\alpha_s = -0.0134 \pm 0.0090$. Even though this value is not statistically significant we can use it in order to obtain some values of the parameter f , just the way we did with n_s . Here, it is found that for $N = 60$ and $N = 70$ we get $f \simeq 0.0133$ and $f \simeq 0.0114$, respectively. These values for the parameter f give rise to $\beta \simeq 297$ and $\beta \simeq 347$, respectively. Although these values satisfy the constraint $\beta > 4N$ new report on the measurement of α_s could change this conclusion, since the value of the parameter f is very sensitive to the change of the value of the parameter α_s .

On the other hand, the tensor spectral index becomes

$$n_T(N) = - \left(\frac{2}{1 - \frac{4N}{\beta}} \right) \left[1 + \left(1 - \frac{1 + \mathcal{C}}{\beta} \right) \left(\frac{1}{1 - \frac{4N}{\beta}} \right) \right], \quad (33)$$

For a single-field inflationary model results a relation at first order between the parameters r and n_T , the so called *consistency relation*, which is expressed by $n_T = -\frac{r}{8}$ [8]. This relationship is due to both scalar and tensor perturbations come from a single degree of freedom, which is carried by the inflaton field. In the second order approximation this consistency condition becomes more subtle[49], but in order to get an estimate is sufficient to consider the above relation. If we use the value reported recently by BICEP2[28] for the r parameter, i.e. $r \simeq 0.2$, it is obtained ruffly $n_T \simeq -0.025$. With this value at hand we get from expression (33) that the values for the parameter f at 60 and 70 e-folds become $f \simeq 0.0319$ (corresponding to $\beta \simeq 121$) and $f \simeq 0.0275$ (corresponding to $\beta \simeq 141$), respectively. Similarly, what happens to the n_s parameter, we find that the inequality $\beta > 4N$ is not met again.

With respect to the running tensor spectral index we find that it becomes in this case

$$\alpha_T(N) = \left(\frac{2}{N}\right) \left[\frac{1}{8}(\beta + 2)(1 + \mathcal{C}) - 1\right] \left(\frac{1}{1 - \frac{4N}{\beta}}\right) \quad (34)$$

Unfortunately, the running tensor index is poorly constrained with the current data set, and thus, usually it is ignored and people constrain the different models using as observable n_s , α_s and r as free parameters. It is expected that upcoming experimental data verify whether they actually are sizable, both tensor mode perturbations and the running of the spectral index. Certainly, this will shed light on the natural structure of the inflaton scalar potential.

Fig 1 represents α_s as a function of the parameter f . It shows that the value $f \simeq 0.425$, or equivalent $\beta \simeq 5.4$, is a value which divides what would correspond to a redshift ($\beta < 5.4$), with α_T negative, and a blueshift ($\beta > 5.4$), with α_T positive. Since β should be greater that $4N$, we see that this model favors a blueshift for the tensor spectral index. However, following a procedure similar to that used in obtaining the consistency condition, it is possible to fix the running tensor spectral index at second order in terms of the r and n_s parameters so that it results into $\alpha_T = \frac{r}{8} \left(\frac{r}{8} + n_s - 1\right)$ [49]. From this expression we could obtain an estimation for the running tensor spectral index if we use those values given by Planck ($n_s \simeq 0.9603$) and BICEP2 ($r \simeq 0.2$). This estimation results to be $\alpha_T \simeq -0.0004$, which, according to the graph 1, it corresponds to a redshift, and thus, violating the inequality $\beta > 4N$.

Let us now consider the tensor-to-scalar ratio r . For intermediate inflation it becomes in terms of the number of e-folding

$$r(N) = \left(\frac{16}{1 - \frac{4N}{\beta}}\right) \left[1 - \frac{4\mathcal{C}}{\beta} \left(\frac{1}{1 - \frac{4N}{\beta}}\right)\right]. \quad (35)$$

Now, from expression (31) we could get N as a function of $1 - n_s$ and then replace this result

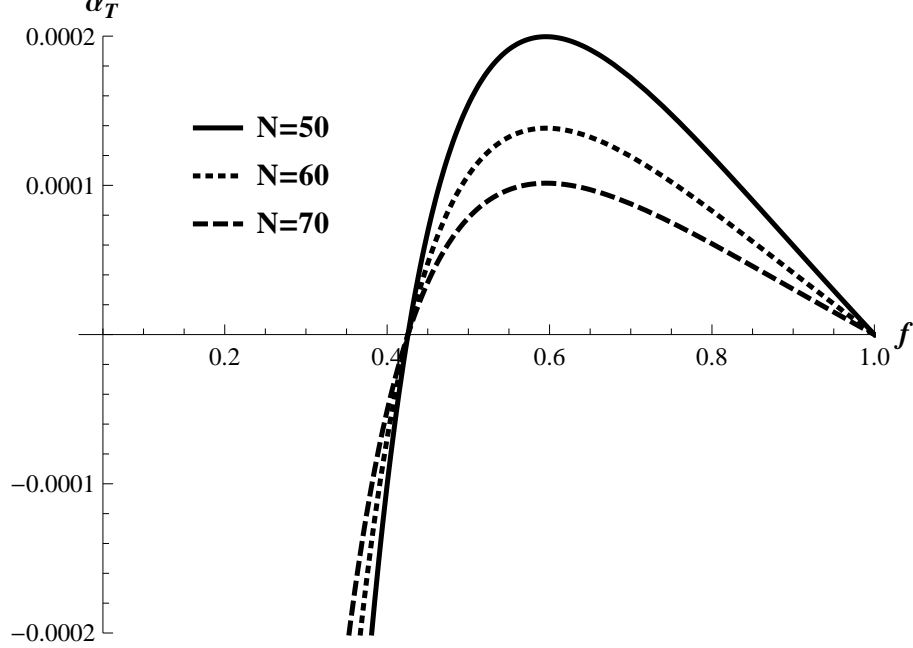


FIG. 1: This plot shows the parameter α_T as a function of the free parameter f for three values of the number of e-folds, $N = 50$, $N = 60$ and $N = 70$.

into the previous expression, Eq. (35), and thus to obtain the parameter r as a function of n_s . This expression reduces to

$$r(n_s)^\pm = \left(\frac{8}{\delta}\right) \left(1 - \frac{2}{\beta}\right) \left[1 \pm \sqrt{1 - \frac{2\delta}{\left(1 - \frac{2}{\beta}\right)^2} (1 - n_s)} \right] \times \left\{ 1 - \frac{2C}{\beta\delta} \left(1 - \frac{2}{\beta}\right) \left[1 \pm \sqrt{1 - \frac{2\delta}{\left(1 - \frac{2}{\beta}\right)^2} (1 - n_s)} \right] \right\}, \quad (36)$$

where $\delta \equiv 1 - \frac{3+C}{\beta} - \frac{2(3-C)}{\beta^2}$. Here, with the finality of obtaining $r = 0$ when $n_s = 1$ we choose in the following the minus sign. With this choice, it is not hard to see that for $\frac{2\delta}{\left(1 - \frac{2}{\beta}\right)^2} (1 - n_s) \ll 1$ (which is equivalent to consider just the first order approximation in the Hubble hierarchy parameters) it is obtained that[50]

$$r(n_s) \approx \frac{8\beta}{(\beta - 2)} (1 - n_s). \quad (37)$$

In Fig. 2 we show the relation between r and n_s for four different values of the parameter f , according to expression (36) (by taking the negative sign), whose curves are correlated with existing measures made by BICEP2[28], together with the data released by Planck[27] combined with the WMAP large-angle polarization. The n_s/r plane showed here corresponds to Fig.13 of Ref.[28], in which the red contours are the Monte Carlo Markov Chains provided with the Planck data release,

while the blue one correspond to the graphs when the BICEP2 data are taken into account. Each of both contours represent the 68% and 95% confidence regions for the corresponding parameters. We notice from this graph that the curves which are in the range $0.7 \lesssim f < 1.0$ lie outside the contour of 68% confidence level. But, with respect to the 95% confidence level region this model is disfavoured, being outside of this contour for any value of the parameter f , or equivalently β .

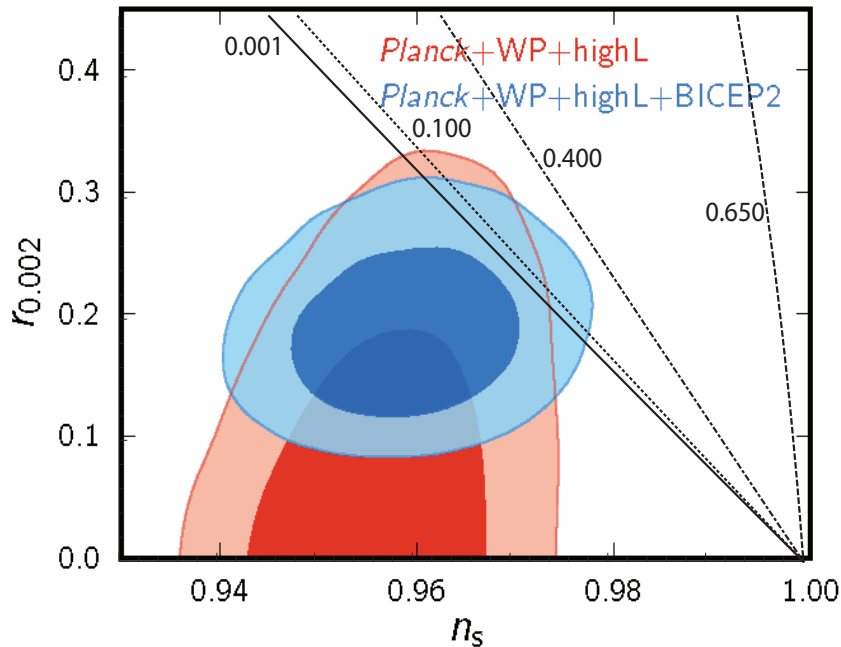


FIG. 2: This plot shows the parameter r as a function of the scalar spectral index n_s for four values of the constant f , which correspond to $f = 0.001; 0.100; 0.400; 0.650$, as described by Eq. (36) by choosing the minus sign. These curves are contrasted with the recent data released by Planck[27] combined with the WMAP large-angle polarization and BICEP2[28]. The n_s/r plane showed here corresponds to Fig.13 from Ref.[28], in which the red contours are the Monte Carlo Markov Chains provided with the Planck data release, while the blue one correspond to the plots when the BICEP2 data are taken into account. Each of both contours represent the 68% and 95% confidence regions. Note that this model is disfavoured, being outside of the 95% contour region for any value of the parameter f , or equivalently β .

VI. CONCLUSION

We have studied single-field model of intermediate inflationary universes. This study was made under the scheme of the Hamilton-Jacobi approach, which mainly rests on the generating function, which is given by the Hubble factor, H , as a function of the scalar inflaton field, ϕ . Within this scheme a series of hierarchical parameters are defined in terms of the generating function and its

derivatives, the so-called hierarchy Hubble parameters. Now, these hierarchy parameters satisfy the set of flow equations, commonly called the hierarchy flow equations. This set of equation corresponds to a set of lineal differential equations, which are characterized by derivatives of the hierarchy Hubble parameters with respect to the number of e-folds. By their nature, all higher-level hierarchy Hubble parameters are determined by the basic parameter, the first hierarchy Hubble parameter and its derivatives, when it is expressed in terms of the number of e-folds.

With the different hierarchy Hubble parameters at hand, we could get at second order in these parameters the corresponding observational cosmological parameters, such that the scalar spectral index, its running and the tensor-to-scalar ratio, among others, related to intermediate inflation. When these parameters are confronted with the current observational data, such that Planck and BICEP2, it is found that this model is disfavoured, since, for instance, in the n_s/r plane all the curves are outside of the 95% contour region for any value of the parameter f , or equivalently β .

Another drawback that this model presents is related to its scalar potential. The main characteristic that the potential presents in this model, it does not have a minimum (except for $\phi \rightarrow \infty$, where $V(\phi) \rightarrow 0$, for any value of β). This makes it impossible for inflation end without the aid of an additional scalar field. In this respect, some authors have introduced the so-called curvaton scalar field.

Before concluding we would like to mention that the intermediate inflationary universe model is similar to other models in which the scale factor lies between a power law and a de-Sitter phase, such as the logamediate scenario, for example[51, 52]. We guess that these sort of models might suffer of the same problems than that found in the intermediate scenario when they are study under the single-field approach. However, we believe that this is a point that should be considered carefully.

Acknowledgments

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